Interaction between birds and Wind turbines

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Mathematics in Industry Study Group Workshop 2016

January 9, 2016

Overview

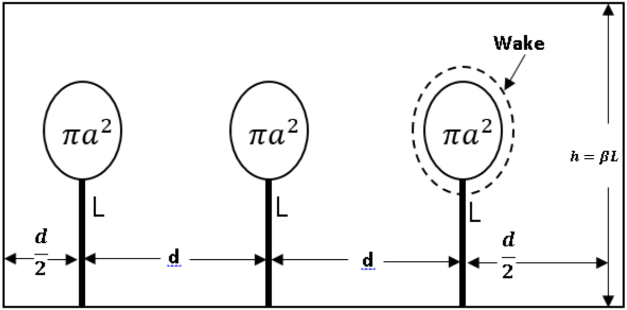
Bird Population Dynamics

$$\frac{dP}{dt} = rP\left(1 - \frac{p}{k}\right)\left(\frac{p}{\alpha} - 1\right) - \sigma P$$

P = population (number of birds)
t = time
r = net growth rate
k = carrying capacity
α = Allee effect (minimum pop before extinction)
σ = probability of a hit by one bird per unit time

The collision risk area

Describing the first row of wind turbines



Mathematical model of σ

$$\sigma = \frac{\text{Area of wind turbines}}{\text{Total area}} = \frac{\pi a^2}{hd}$$

a = length of blade
d = distance between 2 turbines in first row
h = β L = maximum height birds fly above ground
$\beta > 0$ is the height parameter

Equilibrium points

$$rP\left(1-\frac{P}{k}\right)\left(\frac{P}{\alpha}\right)-\sigma P=0$$

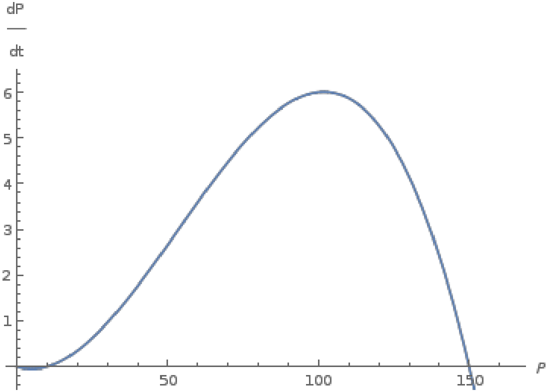
Approximate solutions

$$P=0$$

$$P_{+}=k-\left(\frac{\alpha k}{k-\alpha}\right)\frac{\sigma}{r}+O\left(\frac{\sigma}{r}\right)^{2}$$

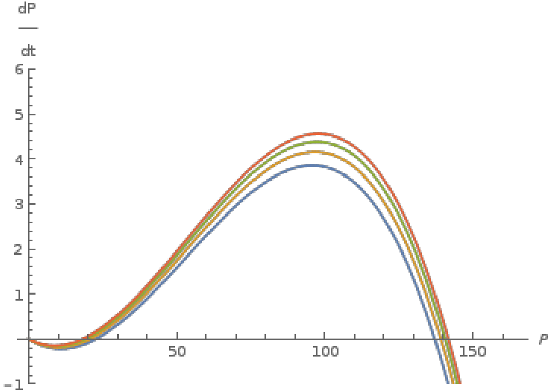
$$P_{-}=\alpha+\left(\frac{\alpha k}{k-\alpha}\right)\frac{\sigma}{r}+O\left(\frac{\sigma}{r}\right)^{2}$$

Graph of $\frac{dP}{dt}$ against P depicting α and k bounds



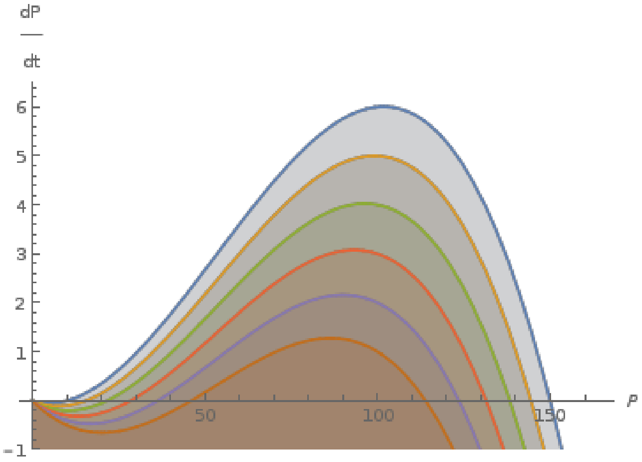
Shows that, absent of sigma, the population change rate is bounded by Allee effect and Carrying Capacity, k

Graph of $\frac{dP}{dt}$ against P showing effect of varying β



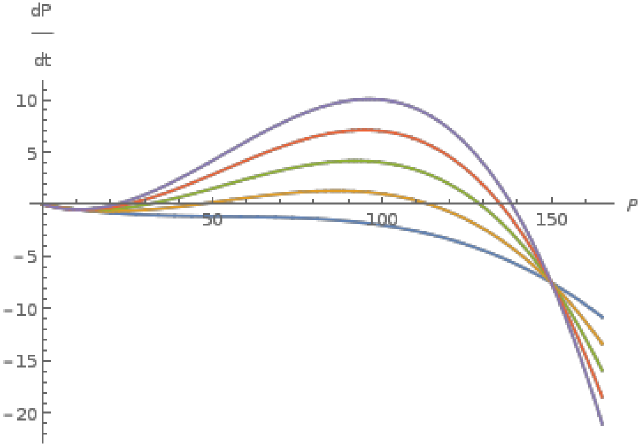
Shows the effect on the population change rate with β increasing in steps of 5 from 5 to 20

Graph of $\frac{dP}{dt}$ against P showing effect of σ on α and k



Shows the impact of increasing values of sigma from 0.01 to 0.05 in steps of 0.01 on the Allee effect and Carrying capacity

Graph of $\frac{dP}{dt}$ against P showing effect of varying growth rate



Shows the impact of increasing net growth rate on Population change rate

Dynamics of the wake the wind turbine

Consider a turbulent wake with radial component of velocity

$$\bar{v}_r = 0 + \bar{v}_r(r, z)$$

and the velocity in the z component (horizontal direction) is

$$\bar{v}_z = U - \bar{w}(r, z)$$

U = incoming wind velocity
$\overline{w}(r, z)$ = the velocity deficit

$\bar{v}_z$ and $\bar{v}_r$ = the time average over fluctuation of v_z and v_r respectively.
We assume that the fluid flow (the wind) is axisymmetric i.e.

$$\frac{\partial}{\partial \theta} = 0$$
$$\bar{v}_\theta = 0$$

Next, we manipulate the Navier-Stokes equations with a boundary layer approximation to retrieve the equation for the upper half of a wake:

$$\bar{v}_r \frac{\partial \bar{v}_r}{\partial r} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left[r \left(\nu + \mathfrak{l}^2(z) \frac{\partial \bar{v}_z}{\partial r} \right) \right] = 0 \qquad (1)$$

where ν = kinematic viscosity, a property of the fluid
$\mathfrak{l}(z)$ = Prandtl's mixing length
$\mathfrak{l}^2(z)\frac{\partial \bar{v}_z}{\partial r}$ = kinematic eddy viscosity due to the turbulence in the wake.

Equation of conservation of mass, for an incompressible fluid:

$$\frac{\partial}{\partial r}(r\bar{v}_r) + r\frac{\partial \bar{v}_z}{\partial z} = 0 \qquad (2)$$

Equations 1 and 2 are substituted into Equation 3 and 4. We make the following assumptions:
1) We know $v_r =$ small therefore, second order terms like $\bar{v}_r\frac{\partial \bar{v}_r}{\partial r}$ can be ignored because they too are small
2) We choose $\mathfrak{l}(z) = \mathfrak{l}_0 z^n$ where n is not specified

Equation 1 simplifies to:

$$U\frac{\partial w}{\partial z} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) - l_0^2 z^{2n}\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial w}{\partial r}\right)^2\right] \tag{3}$$

and Equation 2 simplifies to:

$$r\frac{\partial w}{\partial r} + v_r - r\frac{\partial w}{\partial z} = 0 \tag{4}$$

Equations 3 and 4 are PDEs that are difficult to solve. To reduce them to ODEs, we consider the scaling transformation under which the PDE is invariant:

$$\bar{r} = \lambda^a r$$
$$\bar{z} = \lambda^b z$$
$$\bar{w} = \lambda^c w$$
$$\bar{v}_r = \lambda^d v_r$$

Four unknowns from the scaling transformation, we have three equations for from which we can calculate them from one unknown:

$$\begin{aligned} b &= 2a \\ c &= a(1-4n) \\ d &= -4na \end{aligned}$$

Suppose that:

$$w = f(r,z) \text{is some function that is a solution of Equation 3}$$

then

$$\bar{w} = f(\bar{r},\bar{z}) \text{is also a solution} \tag{5}$$

The characteristic curves for the first order linear PDEs, reduce to:

$$f = z^{\frac{c}{b}} F(\frac{r}{z^{\alpha}})$$

$$g(r,z) = z^{\frac{d}{b}} G(\frac{r}{z^{\alpha}})$$

In summary, the similarity solutions are of the form:

$$w(r,z) = z^{\frac{1}{2}(1-4n)} F(\xi) \tag{6}$$

$$\bar{v}_r = z^{-2n} G(\xi) \quad \text{where } \xi = \frac{r}{z^{\frac{1}{2}}} \tag{7}$$

We get 2 ODEs:

$$l_0^2 \frac{d}{d\xi}\left[\xi(\frac{dF}{d\xi})^2\right] - r\frac{d}{d\xi}\left[\xi\frac{dF}{d\xi}\right] + \frac{1}{2}U\left[(1-4n)\xi F - \xi^2\frac{dF}{d\xi}\right] = 0 \tag{8}$$

$$\frac{d}{d\xi}(\xi G) + \frac{1}{2}\left[\xi^2\frac{dF}{d\xi} - (1-4n)\xi F(\xi)\right] = 0 \tag{9}$$

The boundary conditions:
at r = 0
$\frac{\partial w}{\partial r}(0, z) = 0$
$\bar{v}_r(0, z) = 0$
The boundary conditions:
at r = b(z)
$w\left(b(z), z\right) = 0$
$\frac{\partial w}{\partial r}\left(b(z), z\right) = 0$
(the eddy viscosity vanishes so no turbulence)

Integrating the Equation 3 wrt r

$$\int_0^{b(z)} rw(r,z)dr = \text{constant that is INDEPENDENT of } z$$

The total momentum deficit in the wake is equal to the drag on the air due to the turbine. Using the above equation:

$$D = 2\pi\rho \int_0^{b(z)} rw(r,z)dr = \text{constant that is INDEPENDENT of } z \quad (10)$$

D is a conserved quantity

We have

$$n = \frac{3}{4}$$

$$b(z) = b_0\sqrt{z}$$

the boundary resembles a square root function if the boundary is finite

$$D = 2\pi\rho \int_0^{b(z)} \xi F(\xi) d\xi \quad \text{which is a conserved quantity}$$

$$F(\xi) = \left[B - \frac{1}{3} \left(\frac{U}{2} \right)^{\frac{1}{2}} \frac{1}{l_0} \xi^{\frac{3}{2}} \right]^2 \quad \text{where } B \text{ is the constant of integration} \tag{11}$$

When $r \to \infty$, $\xi \to \infty$, $F \to \infty$ making $w \to \infty$ which is not possible. Therefore, the boundary of the wake is indeed finite. Therefore:

$$F(b_0) = 0$$

We find the equations for $F(\xi)$ and $G(\xi)$:

$$F(\xi) = \frac{1}{18}\frac{U}{\mathfrak{l}_0^2}\left[b_0^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}\right]^2 \tag{12}$$

$$G(\xi) = -\frac{U\xi}{36\mathfrak{l}_0^2}\left[b_0^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}\right]^2 \tag{13}$$

The value for b_0

$$b_0 = \left[\frac{70\, \mathfrak{l}_0^5\, D}{\pi \rho U}\right]^{\frac{1}{5}} \tag{14}$$

$$\begin{aligned} v_z(r,z) &= U - \frac{1}{z}\frac{U}{18\, \mathfrak{l}_0^2}\left[b_0^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}\right]^2 \\ &= U\left[1 - \frac{1}{z\, 18\, \mathfrak{l}_0^2}\left[b_0^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}\right]^2\right] \end{aligned}$$

where the square bracket is the factor by which the speed decreases as it approaches the next turbine

Lastly, we check the sign of $v_r(r,z)$:

$$v_r(r,z) = \frac{1}{z^{\frac{3}{2}}} G(\xi)$$
$$= -\frac{1}{z^{\frac{3}{2}}} \left[\frac{U\xi}{36\, l_0^2} \left[b_0^{\frac{3}{2}} - \xi_0^{\frac{3}{2}} \right]^2 \right]$$

which is negative

Interpretation of results

First model in literature for bird population dynamics factoring in effects of wind turbines:

$$\frac{dP}{dt} = rP\left(1 - \frac{p}{k}\right)\left(\frac{p}{\alpha} - 1\right) - \sigma P$$

$$\sigma = \frac{\text{Area of wind turbines}}{\text{Total area}} = \frac{\pi a^2}{hd} \tag{15}$$

Derived velocity decrease factor between turbines in downstream direction:

$$v_z(r, z) = U\left[1 - \frac{1}{z\,18\,l_0^2}\left[b_0^{\frac{3}{2}} - \xi_0^{\frac{3}{2}}\right]^2\right]$$

where the square bracket is the factor by which the speed decreases as it approaches the next turbine

Derived equation for wake boundary:

$$r = b_0\sqrt{z}$$
$$= \left[\frac{70\, l_0^5\, D}{\pi\rho U}\right]^{\frac{1}{5}} \sqrt{z}$$

Analytically proved that alpha increases and k reduces with an increase in sigma:

Analytically proved that z decreases at a high rate, depicting the area around the turbine where the wake has effect on the flying birds:

$$v_r(r,z) = -\frac{U}{36\, l_0^2}\left[\frac{r}{z^2}\left[b^{\frac{3}{2}} - \xi^{\frac{3}{2}}\right]^2\right]$$

Recommendations for Future Research

Inclusion of avoidance and environmental factors in population dynamics model

Find out turbine placements in the downstream direction wrt distance between them which optimizes generated power and minimizes bird casualties.

Thank you. Questions?

Acknowledgements

Huge and special thanks to Professor Mason